The structure of the jet-stream in a rotating fluid with a horizontal temperature gradient

By RUTH H. ROGERS
Department of Mathematics, University of Manchester*

(Received 28 June 1958)

The jet-stream in a rotating fluid is treated as a thermal boundary layer, but viscous effects are omitted from the first approximation. A theoretical justification for this treatment is presented, and a particular solution of the resulting equations is found. This solution is shown to give a reasonable picture of the flow in the neighbourhood of the stream far from solid boundaries.

1. Introduction

The problem of the convection of a rotating fluid subject to differential heating has been considered in recent years by workers in two fields. Meteorologists are concerned with a rotating atmosphere heated from below: heat is supplied at the equator, and is removed to outer space via the poles. Geophysicists are concerned with the motion of the earth's core, which also rotates and is assumed to be differentially heated. Both groups of workers have devised laboratory experiments to investigate the problem more thoroughly, and although the method of heating varies, the phenomena observed are essentially the same. The experiments have been described by Hide (1956, 1958), Fultz (1951) and Riehl & Fultz (1957).

In each case, the motion relative to the rotating apparatus is observed to be symmetrical about the axis of rotation (which is vertical) for sufficiently high Rossby numbers. The appropriate parameter, which is related to a Rossby number, has been expressed by Hide in the form†

$$\Theta = \frac{gd}{4\omega^2(b-a)^2}\frac{\Delta\rho}{\rho_0}, \tag{1}$$

where g is the acceleration due to gravity, d is the depth of the working fluid, ω is the angular velocity of the apparatus, a and b are the radii of the containing cylinders, $\Delta\rho$ is the applied horizontal density difference across the cylinders, and ρ_0 is the mean density of the fluid. He finds that the flow is symmetrical when Θ is greater than 0·4. When Θ is less than its critical value, the motion becomes asymmetric relative to the apparatus: most of the motion is concentrated into a narrow stream which meanders from boundary to boundary forming a number of equal lobes. A picture of the surface of a three-lobe formation obtained in experiments by the author at Manchester is shown in figure 1 (plate 1). By analogy with

* Now at Royal Aircraft Establishment, Farnborough, Hants.

† Hide defines Θ as four times the value given in (1).

similar narrow streams sometimes observed in the atmosphere, this is usually called the 'jet-stream'. There is some doubt as to the accuracy of this analogy, because there are so many more variables in the atmospheric motions than in the experimental flows (for example, in the atmosphere the Coriolis parameter varies, and there are considerable surface irregularities not present in the laboratory experiment). It seems likely, however, that some of the atmospheric jet-streams have similar causes to the experimental ones. There are slow circulations in the remainder of the fluid, but these are very small. Similar streams have been obtained by Fultz, although his method of heating is slightly different from that of Hide; he supplies heat from below the fluid, in an attempt to simulate more closely the conditions in the atmosphere.

The theory of the symmetrical mode of flow has been studied in detail by Davies (1953), by Lance & DeLand (1957), and by Lance (1958). The instability of this flow has been investigated by Davies (1956), Chandrasekhar (1953) and Kuo (1954). But no completely successful attempt has been made to solve the problem of the existence of the jet-stream itself. This paper is an attempt to find some simplifying assumptions which may help towards a solution of this problem, but does not claim to do more than suggest an approach.

The equations of motion are first put into a non-dimensional form containing the parameter Θ. Approximations are made by taking Θ to be small, and then expanding in powers of Θ. A particular solution of the first approximation is found in the form of a jet-stream, and this can be valid in the stream itself except where it meets the boundaries. The vertical velocity vanishes in the first approximation, but is given in the second. It is also shown that the equations of motion of the atmosphere can be reduced to a form identical with that discussed here, so that a similar solution exists in this case.

2. The equations of motion in the laboratory experiment

The liquid in which the motion takes place is assumed to have a constant coefficient of thermal expansion, α_0, so that its equation of state is given by

$$\rho = \rho_0 - \alpha_0(T - T_0), \tag{2}$$

where ρ is the density corresponding to the temperature T, and ρ_0, T_0 are reference density and temperature. This is a good approximation for all liquids except water near its freezing-point.

It is contained between coaxial cylinders of radii a and b, and has a depth d; the upper surface is free. The common axis of the cylinders is vertical, and the apparatus rotates with angular velocity ω about this axis. We choose a right-handed system of Cartesian axes $Oxyz$, which rotates with the apparatus; Ox is chosen to be along the surface jet-stream, and Oz to be vertically upwards. The sense of Ox is the same as the direction of rotation. A more precise specification of the axes will be given later: here we have only that the direction of motion is roughly along Ox, and that density and velocity gradients are essentially parallel to Oy. This is sufficient to allow us to make a boundary-layer type of approximation later.

It is easily shown that, in a liquid, the effect of variable density enters the equations of motion only in the buoyancy term. In the horizontal momentum equations, and in the equation of continuity, then, we can replace ρ by its mean value, which we choose to be the reference density ρ_0 of equation (2).

The effect of rotation is evident in the Coriolis and centrifugal terms in the horizontal momentum equations. Since only low rates of rotation are considered, the centrifugal terms are neglected, as is usual also in the equations of motion of the atmosphere. This is justified by the experimental observation that the surface of the rotating fluid does not assume the form of a paraboloid significantly different from the special case of a horizontal plane, for rotations at which the jet-stream occurs.

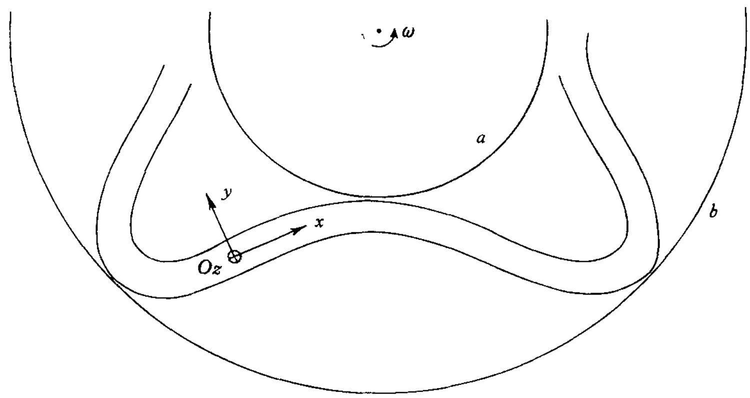

FIGURE 2. The co-ordinate system.

At this stage we can write the equations of horizontal momentum, the hydrostatic equation of vertical equilibrium, the equation of continuity, and the equation of heat transfer in the form

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}-2\omega v = -\frac{1}{\rho_0}\frac{\partial p}{\partial x}+\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right), \tag{3}$$

$$u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}+2\omega u = -\frac{1}{\rho_0}\frac{\partial p}{\partial y}+\nu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right), \tag{4}$$

$$0 = -g-\frac{1}{\rho}\frac{\partial p}{\partial z}, \tag{5}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z} = 0, \tag{6}$$

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}+w\frac{\partial T}{\partial z} = \kappa\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}\right), \tag{7}$$

where p is the pressure at the point (x, y, z), u, v, w are the components of velocity of the fluid parallel to the axes at this point, ν is the kinematic viscosity of the liquid and κ its thermal diffusivity. It is assumed that the motion is steady, and that the frictional dissipation terms in the equation of heat transfer are negligible.

An immediate simplification to these equations can be made by the assumptions of boundary-layer theory. We note that v is very much smaller than u, and that variations of the dependent variables in the direction of Oy are much more rapid than in the directions of Ox and Oz; but we assume that $\partial u/\partial x$ and $\partial v/\partial y$ are of the same order of magnitude. The momentum equations (3) and (4) then reduce to

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}-2\omega v = -\frac{1}{\rho_0}\frac{\partial p}{\partial x}+\nu\frac{\partial^2 u}{\partial y^2}, \tag{8}$$

$$2\omega u = -\frac{1}{\rho_0}\frac{\partial p}{\partial y}, \tag{9}$$

and the heat transfer equation (7) becomes

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}+w\frac{\partial T}{\partial z} = \kappa\frac{\partial^2 T}{\partial y^2}. \tag{10}$$

We can incorporate the equation of state (2) in the hydrostatic equation (5) to give

$$g\rho_0-g\alpha_0(T-T_0) = -\frac{\partial p}{\partial z}. \tag{11}$$

The equations (6), (8), (9) and (11) can now be rearranged in the form

$$g\alpha_0 T = g(\rho_0+\alpha_0 T_0)+\frac{\partial p}{\partial z}, \tag{12}$$

$$u = -\frac{1}{2\omega\rho_0}\frac{\partial p}{\partial y}, \tag{13}$$

$$v = \frac{1}{2\omega\rho_0}\frac{\partial p}{\partial x}+\frac{\dfrac{1}{8\omega^3\rho_0^2}\left(\dfrac{\partial p}{\partial y}\dfrac{\partial^2 p}{\partial x\,\partial y}-\dfrac{\partial p}{\partial x}\dfrac{\partial^2 p}{\partial y^2}\right)+\dfrac{\nu}{4\omega^2\rho_0}\dfrac{\partial^3 p}{\partial y^3}-\dfrac{w}{4\omega^2\rho_0}\dfrac{\partial^2 p}{\partial y\,\partial z}}{1+\dfrac{1}{4\omega^2\rho_0}\dfrac{\partial^2 p}{\partial y^2}}, \tag{14}$$

$$\frac{\partial w}{\partial z} = -\frac{\partial}{\partial y}\left[\frac{\dfrac{1}{8\omega^3\rho_0^2}\left(\dfrac{\partial p}{\partial y}\dfrac{\partial^2 p}{\partial x\,\partial y}-\dfrac{\partial p}{\partial x}\dfrac{\partial^2 p}{\partial y^2}\right)+\dfrac{\nu}{4\omega^2\rho_0}\dfrac{\partial^3 p}{\partial y^3}-\dfrac{w}{4\omega^2\rho_0}\dfrac{\partial^2 p}{\partial y\,\partial z}}{1+\dfrac{1}{4\omega^2\rho_0}\dfrac{\partial^3 p}{\partial y^3}}\right]. \tag{15}$$

Formally, the problem is now solved: for (12) and (13) give T and u in terms of p, (15) can be solved to give w in terms of p, and then (14) used to give v in terms of p. It then remains only to substitute these expressions for T, u, v, w in the equation (10), and solve the resulting equation in p. Unfortunately, this is intractable, and we have to simplify further.

We look for further simplifying assumptions by defining non-dimensional co-ordinates (X, Y, Z), velocities (U, V, W), pressure P and temperature τ, as follows. We take first

$$x = k(b-a)X, \quad y = (b-a)Y, \quad z = dZ, \tag{16}$$

where k is a scale factor to be defined more precisely later; here we note only that k is very much greater than unity. Then we take for the velocities

$$u = 2\omega(b-a)U, \quad v = \frac{2\omega(b-a)}{k}V, \quad w = \frac{2\omega d}{k'}W, \tag{17}$$

where k' is another scale factor; k occurs in the denominator of v, so that $\partial u/\partial x$ and $\partial v/\partial y$ are of the same order. Finally, we take

$$p = -g(\rho_0 + \alpha_0 T_0) z + (gd\Delta\rho) P, \quad \alpha_0 T = \Delta\rho\, \tau, \tag{18}$$

where $\Delta\rho$ is the density difference corresponding to the applied horizontal temperature difference ΔT. Substituting in the equations (12) to (15), we obtain

$$\tau = \frac{\partial P}{\partial Z}, \tag{19}$$

$$U = -\Theta \frac{\partial P}{\partial Y}, \tag{20}$$

$$V = \Theta \frac{\partial P}{\partial X} - \frac{k}{k'} \Theta W \frac{\partial^2 P}{\partial Y \partial Z} + \Theta^2 \mathscr{P}, \tag{21}$$

where
$$\mathscr{P} = \frac{\dfrac{\partial P}{\partial Y}\dfrac{\partial^2 P}{\partial X \partial Y} - \dfrac{\partial P}{\partial X}\dfrac{\partial^2 P}{\partial Y^2} + \sigma\left(\dfrac{2\omega\kappa k\rho_0}{gd\Delta\rho}\right)\dfrac{\partial^3 P}{\partial Y^3}}{1 + \Theta \dfrac{\partial^2 P}{\partial Y^2}}, \tag{21a}$$

$$\frac{k}{k'}\frac{\partial W}{\partial Z} = -\Theta^2 \frac{\partial \mathscr{P}}{\partial Y},$$

where $\sigma = \nu/\kappa$ is the Prandtl number of the fluid. Introducing the non-dimensional variables, and substituting for U and V, in (10), we have

$$-\frac{\partial P}{\partial Y}\frac{\partial T}{\partial X} + \frac{\partial T}{\partial Y}\left(\frac{\partial P}{\partial X} - \frac{k}{k'} W \frac{\partial^2 P}{\partial Y \partial Z} + \Theta \mathscr{P}\right) + \frac{k}{k'} W \frac{\partial T}{\partial Z} = \frac{2\omega\kappa k\rho_0}{gd\Delta\rho}\frac{\partial^2 T}{\partial Y^2}. \tag{22}$$

We can now be more specific about the scale factors k and k'. Since k' occurs only in the ratio k/k' when this multiplies W, we can take $k = k'$ without loss of generality. The factor k occurs only in the expression $2\omega\kappa k\rho_0/gd\Delta\rho$. It seems reasonable, then, to define

$$k = \frac{gd\Delta\rho}{2\omega\kappa\rho_0}. \tag{23}$$

This ratio was found to be of the order of 1000 in the experiments carried out in Manchester by the present author, and so satisfies the condition that k is very much greater than unity. This definition implies that the terms on both sides of equation (22) are of equal importance; that is, that transfer of heat occurs both by conduction and convection. It follows immediately from (21) and (21 a), that the viscous terms are of the same order of magnitude as the inertia terms (as long as σ does not differ greatly from unity). If a smaller value of k is chosen so that the conduction term on the right of equation (22) is negligible, that is, if

$$1 \ll k \ll \frac{gd\Delta\rho}{2\omega\kappa\rho_0},$$

then the viscous term in the expression for $\mathscr{P}$ is also negligible compared with the inertia terms, unless σ is much greater than unity. If k is very much larger than the value given in (23), so that the convection terms on the left of equation (22) are negligible, then the temperature field is independent of the velocity field, and viscous terms may be of importance. These are the assumptions made by Davies in his theories of the symmetrical regime and its instability. We now have the simpler expressions for V, $\mathscr{P}$ and $\partial W/\partial Z$, as follows:

$$V = \Theta\frac{\partial P}{\partial X} - \Theta W\frac{\partial^2 P}{\partial Y\partial Z} + \Theta^2\mathscr{P}, \tag{24}$$

where

$$\mathscr{P} = \frac{\dfrac{\partial P}{\partial Y}\dfrac{\partial^2 P}{\partial X\partial Y} - \dfrac{\partial P}{\partial X}\dfrac{\partial^2 P}{\partial Y^2} + \sigma\dfrac{\partial^3 P}{\partial Y^3}}{1+\Theta\,\partial^2 P/\partial Y^2}, \tag{25}$$

$$\frac{\partial W}{\partial Z} = -\Theta^2\frac{\partial \mathscr{P}}{\partial Y}. \tag{26}$$

Equation (22) can be written

$$-\frac{\partial P}{\partial Y}\frac{\partial^2 P}{\partial X\partial Z} + \frac{\partial^2 P}{\partial Y\partial Z}\left(\frac{\partial P}{\partial X} - W\frac{\partial^2 P}{\partial Y\partial Z} + \Theta\mathscr{P}\right) + W\frac{\partial^2 P}{\partial Z^2} = \frac{\partial^3 P}{\partial Y^2\partial Z}. \tag{27}$$

The only parameter entering into this set of equations is Θ, and this is known to be small. It seems reasonable to look for a solution in powers of Θ. We write

$$P = P_0 + \Theta P_1 + \Theta^2 P_2 + \dots,$$

so that

$$\begin{aligned}
\tau &= \tau_0 + \Theta\tau_1 + \Theta^2\tau_2 + \dots,\\
U &= \Theta U_0 + \Theta^2 U_1 + \Theta^3 U_2 + \dots,\\
V &= \Theta V_0 + \Theta^2 V_1 + \Theta^3 V_2 + \dots,\\
W &= \Theta W_0 + \Theta^2 W_1 + \Theta^3 W_2 + \dots.
\end{aligned}$$

These variables satisfy the equations

$$\tau_0 = \frac{\partial P_0}{\partial Z}, \quad \tau_1 = \frac{\partial P_1}{\partial Z}, \quad \tau_2 = \frac{\partial P_2}{\partial Z}, \dots,$$

$$U_0 = -\frac{\partial P_0}{\partial Y}, \quad U_1 = -\frac{\partial P_1}{\partial Y}, \quad U_2 = -\frac{\partial P_2}{\partial Y}, \dots,$$

$$V_0 = \frac{\partial P_0}{\partial X}, \quad V_1 = \frac{\partial P_1}{\partial X} + W_0\frac{\partial^2 P_0}{\partial Y\partial Z} + \mathscr{P}_0, \dots,$$

$$\frac{\partial W_0}{\partial Z} = 0, \quad \frac{\partial W_1}{\partial Z} = -\frac{\partial \mathscr{P}_0}{\partial Y}, \dots,$$

where

$$\mathscr{P}_0 = \frac{\partial P_0}{\partial Y}\frac{\partial^2 P_0}{\partial X\partial Y} - \frac{\partial P_0}{\partial X}\frac{\partial^2 P_0}{\partial Y^2} + \sigma\frac{\partial^3 P_0}{\partial Y^3}.$$

We have from (27) that

$$-\frac{\partial P_0}{\partial Y}\frac{\partial^2 P_0}{\partial X\partial Z} + \frac{\partial P_0}{\partial X}\frac{\partial^2 P_0}{\partial Y\partial Z} + W_0\left(\frac{\partial^2 P_0}{\partial Z^2} - \left(\frac{\partial^2 P_0}{\partial Y\partial Z}\right)^2\right) = \frac{\partial^3 P_0}{\partial Y^2\partial Z}, \tag{28}$$

and more complicated expressions relating W_1 and P_1, etc.

In the first approximation we have

$$\frac{\partial W_0}{\partial Z} = 0 \quad \text{or} \quad W_0 = \text{constant};$$

it follows at once that $W_0 = 0$ everywhere. Equation (28) now simplifies further to

$$-\frac{\partial P_0}{\partial Y}\frac{\partial^2 P_0}{\partial X \partial Z}+\frac{\partial P_0}{\partial X}\frac{\partial^2 P_0}{\partial Y \partial Z} = \frac{\partial^3 P_0}{\partial Y^2 \partial Z}. \tag{29}$$

If this equation can be solved, an expression for $\mathscr{P}_0$ can be obtained, which can be used to find the form of $\partial W_1/\partial Z$ without difficulty. We have

$$\frac{\partial W_1}{\partial Z} = -\frac{\partial}{\partial Y}\left(\frac{\partial P_0}{\partial Y}\frac{\partial^2 P_0}{\partial X \partial Y}-\frac{\partial P_0}{\partial X}\frac{\partial^2 P_0}{\partial Y^2}+\sigma\frac{\partial^3 P_0}{\partial Y^3}\right). \tag{30}$$

It is difficult to define precise boundary conditions at this stage; they will depend to a considerable extent on how much of the motion we try to describe by the above equations. Further discussion of this problem is postponed until a solution of (29) has been found.

T. V. Davies has pointed out that in the first approximation, the equation of heat transfer can be written in the form

$$-\frac{\partial p}{\partial y}\frac{\partial T}{\partial x}+\frac{\partial p}{\partial x}\frac{\partial T}{\partial y} = 2\omega\rho_0\kappa\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right),$$

where the term representing conduction in the vertical is now the only one omitted. This equation is left unchanged by a conformal transformation in the xy-plane. It follows that we lose no generality by taking rectangular axes, and assuming that the stream flows parallel to Ox; for any solution so obtained is still valid after a conformal transformation designed to transform Ox into some other curve.

If the first approximation is used with the equations in cylindrical polar form, and we assume that variations with respect to r are much greater than those with respect to θ, the fundamental equation to be solved for p is

$$-\frac{\partial p}{\partial r}\frac{\partial^2 p}{\partial \theta \partial z}+\frac{\partial p}{\partial \theta}\frac{\partial^2 p}{\partial r \partial z} = 2\omega\rho_0\kappa\frac{\partial}{\partial r}\left(r\frac{\partial^2 p}{\partial r \partial z}\right).$$

Writing $R = \log r$, this reduces to the form

$$-\frac{\partial p}{\partial R}\frac{\partial^2 p}{\partial \theta \partial z}+\frac{\partial p}{\partial \theta}\frac{\partial^2 p}{\partial R \partial z} = 2\omega\rho_0\kappa\frac{\partial^2 p}{\partial R \partial z}, \tag{31}$$

and this is of the same form as in local Cartesian co-ordinates with y replaced by $R = \log r$, and x replaced by θ.

3. A particular solution of the approximate equation

We have essentially to solve equation (29), but it is more convenient to return to physical variables at this stage. In the first approximation, we have to neglect all inertia terms and viscous terms in equation (8), put $w = 0$ in (10) and leave (9) and (11) as they stand. We can eliminate the hydrostatic part of the pressure from the problem by writing

$$p = -g(\rho_0+\alpha_0 T_0)z+p_0. \tag{32}$$

Then (11) becomes

$$\alpha_0 T = \frac{1}{g}\frac{\partial p_0}{\partial z}, \tag{33}$$

and equations (8) and (9) give

$$u = -\frac{1}{2\omega\rho_0}\frac{\partial p_0}{\partial y}, \quad v = \frac{1}{2\omega\rho_0}\frac{\partial p_0}{\partial x}. \tag{34}$$

The equation of continuity (6), is automatically satisfied, since $w = 0$. The heat transfer equation (10) now gives

$$-\frac{\partial p_0}{\partial y}\frac{\partial^2 p_0}{\partial x \partial z} + \frac{\partial p_0}{\partial x}\frac{\partial^2 p_0}{\partial y \partial z} = 2\omega\rho_0\kappa\frac{\partial^3 p_0}{\partial y^2 \partial z}. \tag{35}$$

One of the properties of the jet-stream of the experiments is its approximately constant width. This suggests a solution of the form

$$p_0 = p_1(x, z)\, p_2(y).$$

Such a solution of equation (35) is possible if

$$p_2\frac{dp_2}{dy} \propto \frac{d^2 p_2}{dy^2}.$$

This implies that *either* $p_2(y) = \tan(y/y_0)$, which does not correspond to the physical problem, *or*

$$p_2(y) = \tanh(y/y_0),$$

where y_0 is an arbitrary constant. Taking this value of p_2, (35) becomes

$$-p_1\frac{\partial^2 p_1}{\partial x \partial z} + \frac{\partial p_1}{\partial x}\frac{\partial p_1}{\partial z} = -\frac{4\omega\rho_0\kappa}{y_0}\frac{\partial p_1}{\partial z};$$

the general solution of this equation is

$$p_1 = -\frac{4\omega\rho_0\kappa}{y_0}\frac{F_1(x) + G_1(z)}{F_1'(x)},$$

where $F_1(x)$, $G_1(z)$ are arbitrary functions of x, z respectively. We have, therefore, a solution of the form

$$p_0 = -\frac{4\omega\rho_0\kappa}{y_0}\frac{F_1(x) + G_1(z)}{F_1'(x)}\tanh\frac{y}{y_0}. \tag{36}$$

The corresponding values of T, u, v are given by equations (33) and (34) in the form

$$\alpha_0 T = -\frac{4\omega\rho_0\kappa}{g y_0}\frac{G_1'(z)}{F_1'(x)}\tanh\frac{y}{y_0}, \tag{37}$$

$$u = \frac{2\kappa}{y_0^2}\frac{F_1(x) + G_1(z)}{F_1'(x)}\operatorname{sech}^2\frac{y}{y_0}, \tag{38}$$

$$v = -\frac{2\kappa}{y_0}\left[1 - \frac{F_1''(x)\{F_1(x) + G_1(z)\}}{\{F_1'(x)\}^2}\right]\tanh\frac{y}{y_0}. \tag{39}$$

This is the simplest solution we can find of the equation, and it is worth considering its nature. It certainly describes a jet-stream in the direction of Ox, which is a streamline, an isotherm, and a line of maximum velocity. There is a horizontal temperature gradient across the stream of magnitude

$$\frac{\partial T}{\partial y} = -\frac{4\omega\rho_0\kappa}{g\alpha_0 y_0^2}\frac{G_1'(z)}{F_1'(x)}\operatorname{sech}^2\frac{y}{y_0}, \tag{40}$$

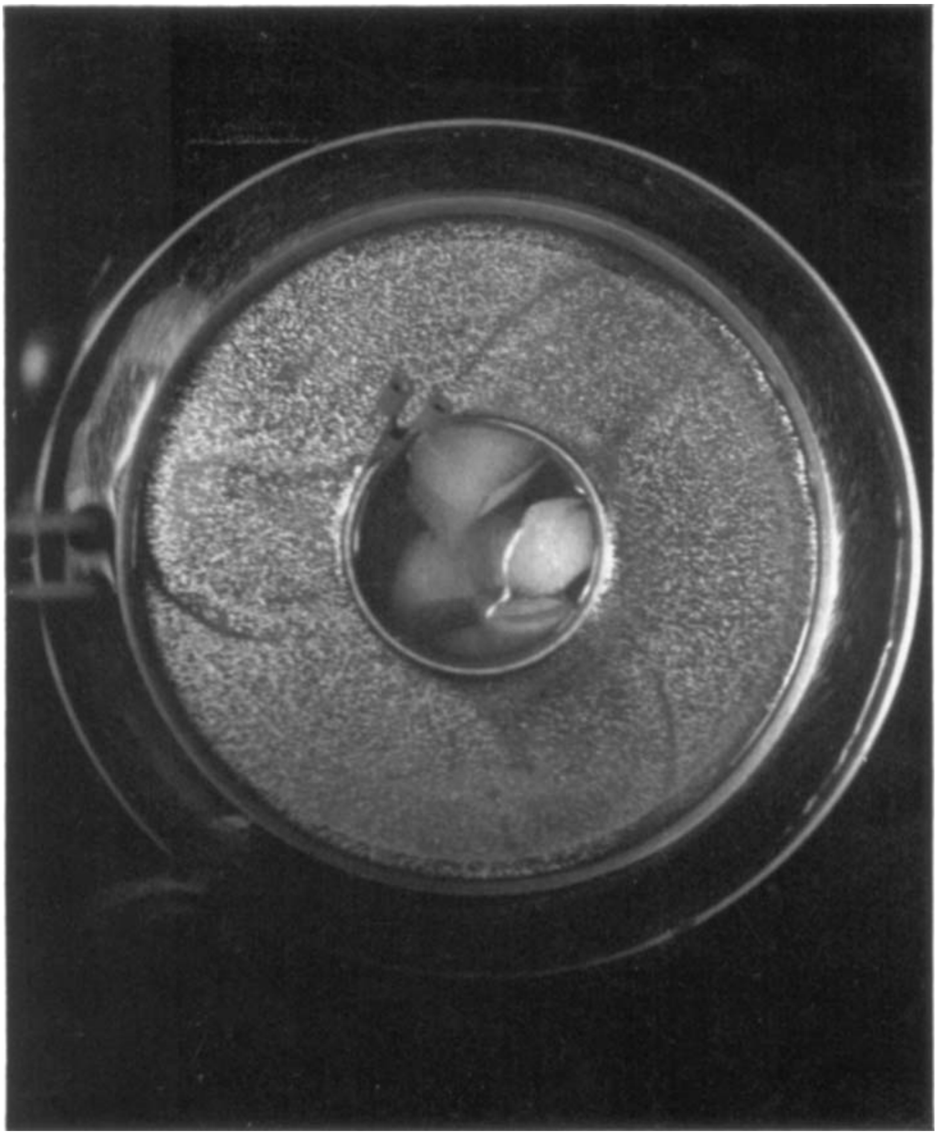

FIGURE 1 (plate 1).

and the temperature itself approaches a limit on each side of the stream. The 'width' of the stream can be taken as $3y_0$, since $\operatorname{sech}^2 3 = 0{\cdot}01$ and $\tanh 3 = 0{\cdot}995$; so, for $|y|$ greater than or equal to $3y_0$, u and $\partial T/\partial y$ are less than 1 % of their values on $y = 0$, and T is within $\frac{1}{2}$ % of its limiting value. The cross-velocity, v, is an odd function of y, so that there is either an inflow towards $y = 0$, or an outflow from $y = 0$; this solution gives no velocity across $y = 0$.

The general functions $F_1(x)$ and $G_1(z)$ are difficult to determine, because of the lack of precise boundary conditions. It is convenient to discuss them in terms of the density difference across the stream

$$(\Delta\rho)' = \frac{8\omega\rho_0\kappa}{gy_0^2}\frac{G_1'(z)}{F_1'(x)}, \tag{41}$$

and the velocity on the x-axis

$$u_0 = \frac{2\kappa}{y_0^2}\frac{F_1(x)+G_1(z)}{F_1'(x)}. \tag{42}$$

The function $G_1'(z)$ can be left arbitrary: in the Chicago experiments of Fultz and his co-workers, where the fluid is heated from below, it is certainly not a constant; in the Cambridge and Manchester experiments, an attempt to make the boundary temperatures independent of z is made, but there are certainly vertical temperature gradients in the fluid. It is of interest, however, to note that there does exist a formal solution for the case in which there are no vertical temperature gradients; in this case, the velocity increases linearly with z. We note that the vertical conduction term $\partial^2T/\partial z^2$ in the heat transfer equation is then identically zero (not merely negligible, as assumed in the derivation).

The function $F_1(x)$ is more troublesome. We note first that $F_1'(x)$ has always the same sign (or is zero), since $(\Delta\rho)'$, as given by (41), cannot change sign with x. It follows that $F_1(x)$ is a monotonic function of x, and cannot be periodic. Hence, either $F_1(x)/F_1'(x)$ or $F_1'(x)$ itself is monotonic, and u cannot be periodic in x as long as it is continuous. Since the jet-stream touches one or other boundary at certain values of x, and there is a sudden temperature change at such a point, it is not unreasonable to obtain discontinuities in the solution.

As a possible example, we consider

$$\frac{F_1(x)}{F_1'(x)} = A + B\cos 2mx \quad (B < A).$$

After integration and some manipulation, we obtain

$$\frac{1}{F_1'(x)} = \frac{A+B\cos 2mx}{C}\,e^{-\theta/m\alpha\beta},$$

where $$\alpha^2 = A+B, \quad \beta^2 = A-B, \quad \tan\theta = (\beta/\alpha)\tan mx.$$

If we always choose $-\frac{1}{2}\pi < \theta < \frac{1}{2}\pi$, then the exponential factor in the expression for $1/F_1'(x)$ is periodic, with a period in x of $\pi/2m$; the $\cos 2mx$ term has a period in x of π/m. As x increases, the exponential term decreases—this corresponds to a decrease in $1/F'(x)$, and hence in $(\Delta\rho)'$, as the stream gets further from a boundary. When it reaches the other boundary, there is a sudden increase in $(\Delta\rho)'$—that is, a discontinuous increase in $1/F_1'(x)$. The factor $(A+B\cos 2mx)$ oscillates about the constant value A, never changing sign; it has twice the period of the exponential factor, and passes through the value A at each discontinuity of θ. The way in which θ varies with x is shown in figure 3.

We can consider one interval of this solution only, taking $-(\pi/2m) < x < (\pi/2m)$, say. If B is very much less than A, so that β is approximately equal to α, we can write $\theta = \beta mx/\alpha$ approximately within the interval; also, to this approximation

$$\frac{F_1(x)}{F_1'(x)} = A, \quad \frac{1}{F_1'(x)} = \frac{A}{C}e^{-x/A}, \quad \frac{F_1''(x)}{F_1'(x)} = \frac{1}{A}.$$

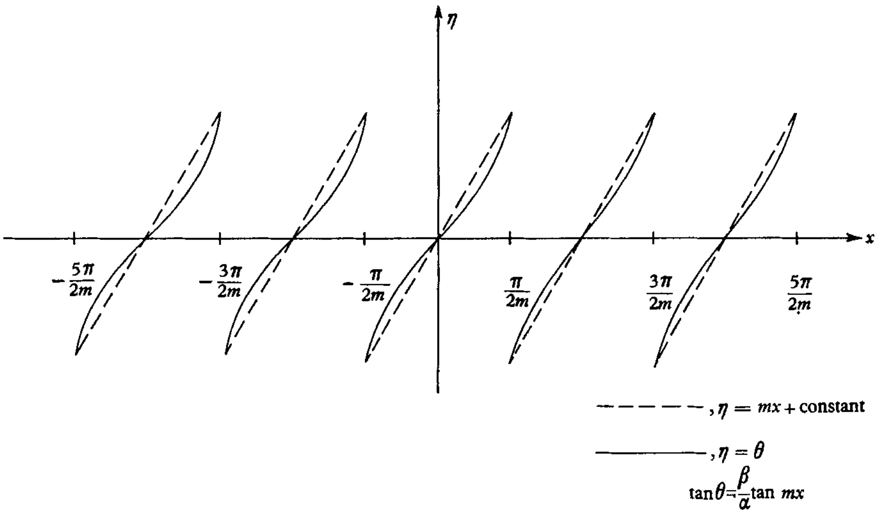

FIGURE 3. The variation of the function θ with x.

This gives, for T, u and v in any interval far from the boundary

$$\left.\begin{aligned} T &= -\frac{4\omega\rho_0\kappa}{\alpha_0 g y_0^2}\frac{A}{C}G_1'(z)\,e^{-x/A}\tanh y/y_0, \\ u &= \frac{2\kappa}{y_0^2}A\left(1+\frac{G_1(z)}{C}e^{-x/A}\right)\operatorname{sech}^2 y/y_0, \\ v &= \frac{2\kappa}{y_0}\frac{G_1(z)}{C}e^{-x/A}\tanh y/y_0. \end{aligned}\right\} \tag{43}$$

Such a solution implies that during the passage from one boundary to another, the temperature difference across the stream, and the velocity u decrease slightly, and that there is an outflow everywhere which also decreases in magnitude as x increases. (In the experiment, there is almost certainly a cross-flow, but this may appear in this solution only in the second approximation.) The approximate particular solution of equations (43) can be fitted into the experimental pattern as shown in figure 4, where the dotted lines indicate the slow circulations which take place outside the jet-stream.

It is not suggested that this is the best form of the function $F_1(x)$, but it is put forward as a possible one; it is probably the simplest. The jet-stream described by this solution does not obviously differ from that observed in a fundamental way, although the mechanism of transition between the jet-stream and the slow circulations described above is not obvious.

We can relate the solution of equation (43) more closely to experiment, if we incorporate the constant C in the arbitrary function G_1, and write

$$\frac{4\omega\rho_0\kappa}{gy_0^2}A = \tfrac{1}{2}\Delta\rho,$$

or

$$A = \frac{gy_0\Delta\rho}{8\omega\rho_0\kappa}y_0. \tag{44}$$

Then we have

$$\alpha_0 T = -\tfrac{1}{2}\Delta\rho\, G_1'(z)\, e^{-x/A}\tanh y/y_0,$$

$$u = \frac{g\Delta\rho}{4\omega\rho_0}(1 + G_1(z)\,e^{-x/A})\,\mathrm{sech}^2\, y/y_0,$$

$$v = \frac{2\kappa}{y_0}G_1(z)\,e^{-x/A}\tanh y/y_0.$$

We note that

$$\frac{u}{v} \sim \frac{gy_0\Delta\rho}{8\omega\kappa\rho_0} = \frac{A}{y_0},$$

and this is to be compared with the scale-factor k defined by equation (23). It is certainly smaller than k, by a factor of $y_0/4d$, but is probably still large enough to justify the retention of the conduction term. A study of the separate terms of the

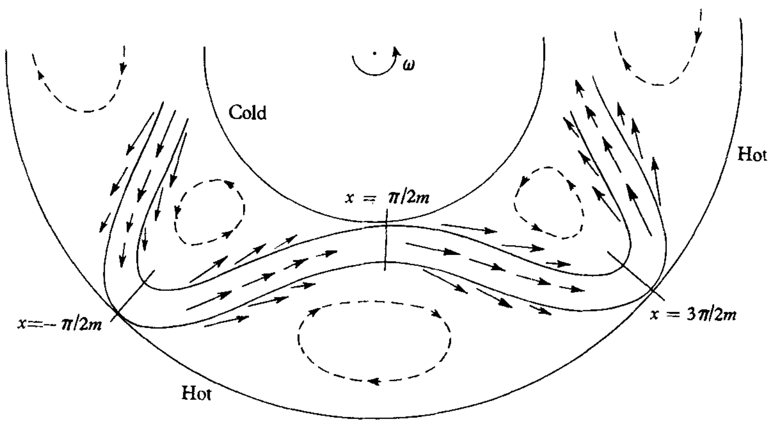

FIGURE 4. A particular solution of the first approximation.

heat-transfer equation indicate that the conduction is balanced entirely by the convection due to the part of u which is independent of z (this is true of the general equation); since u varies considerably with height, it seems likely that this is the smaller term, and this is consistent with a small conduction term.

We can equate A, which is essentially the scale of x, to $\pi(b+a)/n$, where n is the number of lobes in the pattern, and obtain an order of magnitude for the width of the jet-stream, $3y_0$. We have

$$\frac{\pi(b+a)}{n} = \frac{gy_0^2\Delta\rho}{8\omega\rho_0\kappa},$$

or

$$y_0^2 = \frac{8\pi\omega\rho_0\kappa(b+a)}{gn\Delta\rho} \sim 0{\cdot}4\,\mathrm{cm}^2.$$

This gives y_0 of the order 0·6 cm, and the width of the stream of order 2 cm; this is in agreement with experiment.

In cylindrical polar co-ordinates, we can use equation (31) instead of (35), to obtain a solution of the form (37), (38) and (39), with x replaced by θ, and with $\tanh(y/y_0)$ replaced by

$$\tanh(\log r^{\gamma}) = (r^{2\gamma}-1)/(r^{2\gamma}+1),$$

where γ is an arbitrary constant. The necessary discontinuity with respect to θ makes this solution impracticable.

4. A generalization of this solution

By inspection, it is easy to generalize the solution (36) of equation (35) to the form

$$p_0 = -4\omega\rho_0\kappa\left\{\frac{F_1(x)+G_1(z)}{F_1'(x)}\left[\frac{1}{y_0(x)}\tanh\frac{y-y_1(x)}{y_0(x)} - F_2'(x)\right] + F_2(x) + G_2(z)\right\}, \tag{45}$$

where $F_1(x)$, $F_2(x)$, $y_0(x)$, $y_1(x)$ are arbitrary functions of x, and $G_1(z)$, $G_2(z)$ of z. The other variables are then given by

$$\alpha_0 T = -\frac{4\omega\rho_0\kappa}{g}\left\{\frac{G_1'(z)}{F_1'(x)}\left[\frac{1}{y_0(x)}\tanh\frac{y-y_1(x)}{y_0(x)} - F_2'(x)\right] + G_2'(z)\right\}, \tag{46}$$

$$u = \frac{2\kappa}{[y_0(x)]^2}\,\frac{F_1(x)+G_1(z)}{F_1'(x)}\operatorname{sech}^2\frac{y-y_1(x)}{y_0(x)}, \tag{47}$$

$$\begin{aligned} v = &-\frac{2\kappa}{y_0(x)}\left\{1-\frac{F_1''(x)\,[F_1(x)+G_1(z)]}{[F_1'(x)]^2}\right\}\tanh\frac{y-y_1(x)}{y_0(x)} \\ &+2\kappa\frac{F_1(x)+G_1(z)}{F_1'(x)}\left\{\frac{y_0'(x)}{[y_0(x)]^2}\tanh\frac{y-y_1(x)}{y_0(x)} + F_2''(x) - \frac{F_2'(x)\,F_1''(x)}{F_1'(x)}\right. \\ &\left.+\frac{1}{y_0(x)}\left[\frac{y_1'(x)}{y_0(x)} + \frac{\{y-y_1(x)\}\,y_0'(x)}{[y_0(x)]^2}\right]\operatorname{sech}^2\frac{y-y_1(x)}{y_0(x)}\right\}. \end{aligned} \tag{48}$$

We consider the effect of the new functions introduced into the solution, and shall show that most of them can be ignored.

The function $G_2(z)$ does not appear in the expression for the velocity; its effect is, then, to allow an overall variation of temperature (and hence pressure) in the vertical. As with the function $G_1(z)$, it is difficult to find suitable boundary conditions, and it may be left arbitrary. No doubt the form of $G_2(z)$ in experiments where heating is from below differs from that when the applied heating is purely radial. We note that the form of the function $G_2(z)$ does not alter the expression for the density difference $(\Delta\rho)'$ across the stream, as given in (41).

Since the width of the jet-stream is proportional to $y_0(x)$, we can regard this function as applying to a jet-stream of variable width. As stated in the previous section, the width of the stream in the experiments varies very little, and so we can take $y_0(x) = y_0$, a constant. In atmospheric jets, however, it may be necessary to retain the function, as shown in §6.

The jet form of the solution now has its maximum value of u about $y = y_1(x)$, instead of about $y = 0$; since the solution is based on the assumption that the jet is more or less parallel to $y = 0$, it follows that the two curves are nearly parallel. This means that $y_1'(x)$ cannot be large. We can introduce the function, then, to introduce a small curvature of the jet-stream. A large departure from $y = 0$ must be dealt with by means of a conformal transformation, as explained in § 2.

The nature of the function $F_2(x)$ can be seen most readily if we write

$$f'(x) = F_2'(x)/F_1'(x),$$

so that

$$f''(x) = \frac{F_2''(x)}{F_1'(x)} - \frac{F_2'(x)\,F_1''(x)}{[F_1'(x)]^2}.$$

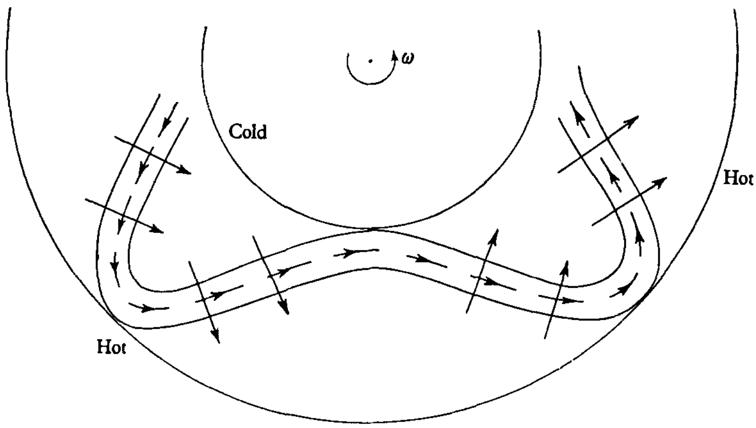

FIGURE 5. The cross-flow in a more general solution of the first approximation.

Then, if y_0 is a constant, we have

$$\alpha_0 T = -\frac{4\omega\rho_0\kappa}{g}\left\{\frac{G_1'(z)}{F_1'(x)}\frac{1}{y_0}\tanh\frac{y-y_1(x)}{y_0} - f'(x)\,G_1'(z) + G_2'(z)\right\}, \tag{49}$$

$$u = \frac{2\kappa}{y_0^2}\frac{F_1(x)+G_1(z)}{F_1'(x)}\operatorname{sech}^2\frac{y-y_1(x)}{y_0}, \tag{50}$$

$$v = -\frac{2\kappa}{y_0}\left\{1 - \frac{F_1''(x)\,[F_1(x)+G_1(z)]}{[F_1'(x)]^2}\right\}\tanh\frac{y-y_1(x)}{y_0}$$
$$+\frac{2\kappa}{y_0^2}\frac{F_1(x)+G_1(z)}{F_1'(x)}y_1'(x)\operatorname{sech}^2\frac{y-y_1(x)}{y_0} + 2\kappa[F_1(x)+G_1(z)]f''(x). \tag{51}$$

We see that the function $f'(x)$ implies an overall variation of T with x, and a cross-flow term in v. As the stream travels from the hot to the cold boundary, we would expect $f'(x)$ to decrease; it follows at once that $f''(x)$ is negative in this case, and a positive value of v results. Conversely, there is a negative value of v as the stream approaches the hot boundary. The cross-flow therefore corresponds to a small circulation of the whole fluid as shown in figure 5. This cross-flow is, of course, superposed on the out-flow or inflow corresponding to the solution of § 3.

We note that on the line $y = y_1(x)$, the slope of the isotherms and that of the streamlines are the same, and are given by the expression

$$\frac{dy}{dx} = y_1'(x) + y_0^2 F_1'(x) f''(x). \tag{52}$$

The line itself is not itself a streamline or an isotherm unless

$$y_1'(x) + y_0^2 F_1'(x) f''(x) = 0.$$

If this relation holds, however, the line is both a streamline and an isotherm, and this does not agree with experiment.

5. The second approximation: the vertical velocity

The variation of the vertical velocity with height is given in terms of the non-dimensional variables by (30); the corresponding relation in terms of the physical variables is

$$\frac{\partial w}{\partial z} = -\frac{1}{8\omega^3\rho_0^2}\frac{\partial}{\partial y}\left(\frac{\partial p_0}{\partial y}\frac{\partial^2 p_0}{\partial y\,\partial x} - \frac{\partial p_0}{\partial x}\frac{\partial^2 p_0}{\partial y^2} + 2\omega\rho_0\nu\frac{\partial^3 p_0}{\partial y^3}\right), \tag{53}$$

where p_0 is given in the first approximation. We can write this in the form

$$\frac{\partial w}{\partial z} = \frac{\partial w_i}{\partial z} + \frac{\partial w_v}{\partial z}, \tag{54}$$

where
$$\frac{\partial w_i}{\partial z} = -\frac{1}{8\omega^3\rho_0}\frac{\partial}{\partial y}\left(\frac{\partial p_0}{\partial y}\frac{\partial^2 p_0}{\partial x\,\partial y} - \frac{\partial p_0}{\partial x}\frac{\partial^2 p_0}{\partial y^2}\right) \tag{54a}$$

is dependent on the inertia terms, and

$$\frac{\partial w_v}{\partial z} = -\frac{\nu}{4\omega^2\rho_0}\frac{\partial^4 p_0}{\partial y^4} \tag{54b}$$

is due to the viscosity. Since this term contains a derivative of p_0 of higher order than the original equation for p_0, it is doubtful whether we can include the viscous effect in this form. An alternative method would be to replace the term $\nu\,\partial^2 u/\partial y^2$ in the x-momentum equation by Ku, where K is a friction constant. In this case we have

$$\frac{\partial w_v}{\partial z} = -\frac{K}{4\omega^2\rho_0}\frac{\partial^2 p_0}{\partial y^2}. \tag{54c}$$

Both forms of the term will be discussed.

Substituting the solution given in equations (50), (51) into (54a), and taking $y_0(x) = y_0$, a constant, we have

$$\begin{aligned}\frac{\partial w_i}{\partial z} = {} & \frac{8\kappa^2}{\omega y_0^5}\frac{F_1(x)+G_1(z)}{F_1'(x)}\left\{1 - \frac{F_1''(x)\,[F_1(x)+G_1(z)]}{[F'(x)]^2}\right\}\operatorname{sech}^2\frac{y-y_1(x)}{y_0}\tanh^3\frac{y-y_1(x)}{y_0} \\ & + \frac{4\kappa^2}{\omega y_0^4}\left[\frac{F_1(x)+G_1(z)}{F_1'(x)}\right]^2\left[F_2''(x) - \frac{F_1''(x)\,F_2'(x)}{F_1'(x)}\right]\left(3\operatorname{sech}^4\frac{y-y_1(x)}{y_0} - 2\operatorname{sech}^2\frac{y-y_1(x)}{y_0}\right).\end{aligned} \tag{55}$$

We note that this can be expressed in the form

$$\frac{\partial w_i}{\partial z} = -\frac{2}{\omega y_0^2} uv \tanh^2 \frac{y - y_1(x)}{y_0} + \frac{3}{\omega} u^2 F_1'(x) f''(x). \tag{55a}$$

This consists of two terms, the first corresponding to the jet part of the solution, and the second to the cross-flow part introduced by the function $F_2(x)$ of §4. The sign of the cross-flow part is therefore negative (suggesting downflow) when the stream approaches a cold boundary, and positive (suggesting upflow) when

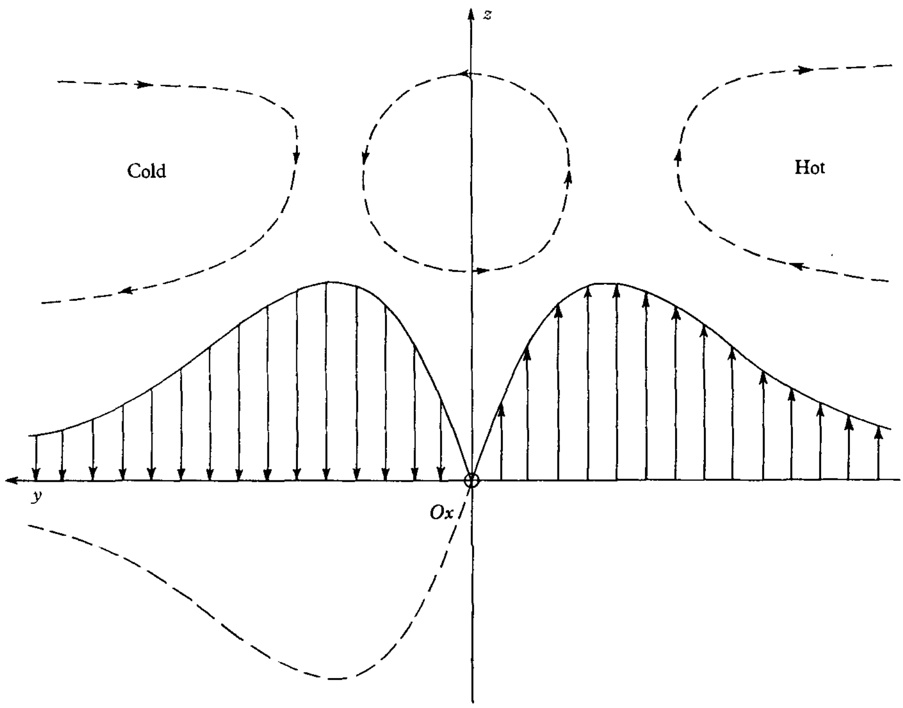

FIGURE 6. The cross-circulation in the second approximation, due to inertia forces.

the stream approaches a hot boundary. Its magnitude is large near $y = y_1(x)$, and falls off rapidly as $|y - y_1(x)|$ increases; it is an even function of $y - y_1(x)$. The jet part of (55*a*) has the same sign as uv; it varies across the stream like

$$-\operatorname{sech}^2 \frac{y - y_1(x)}{y_0} \tanh^3 \frac{y - y_1(x)}{y_0},$$

when there is outflow in the first approximate solution (as in the solution of equation (43)); this gives values for w near $z = 0$ as shown in figure 6. This suggests a cross-circulation as shown by the dotted lines, and this is consistent with the observations of Riehl & Fultz (1957). It is assumed that the velocities given by the second term of (55*a*) are superposed on this circulation, but do not alter it essentially; certainly in the mean with respect to x, they will not do so, as the term changes sign.

The viscous term as given by (54c) becomes

$$\frac{\partial w_v}{\partial z} = -\frac{2K\kappa}{\omega y_0^3}\frac{F_1(x)+G_1(z)}{F_1'(x)}\operatorname{sech}^2\frac{y-y_1(x)}{y_0}\tanh\frac{y-y_1(x)}{y_0}$$
$$= -2\frac{K}{\omega y_0}u\tanh\frac{y-y_1(x)}{y_0}.$$

This gives a cross-circulation similar in form to that of figure 6, though the maximum value of $|\partial w_v/\partial z|$ occurs for a different value of $|y-y_1(x)|$ from that for which $|\partial w_i/\partial z|$ is a maximum. It follows that, if this is the correct form for the viscous terms, the cross-circulation is similar to that given by figure 6.

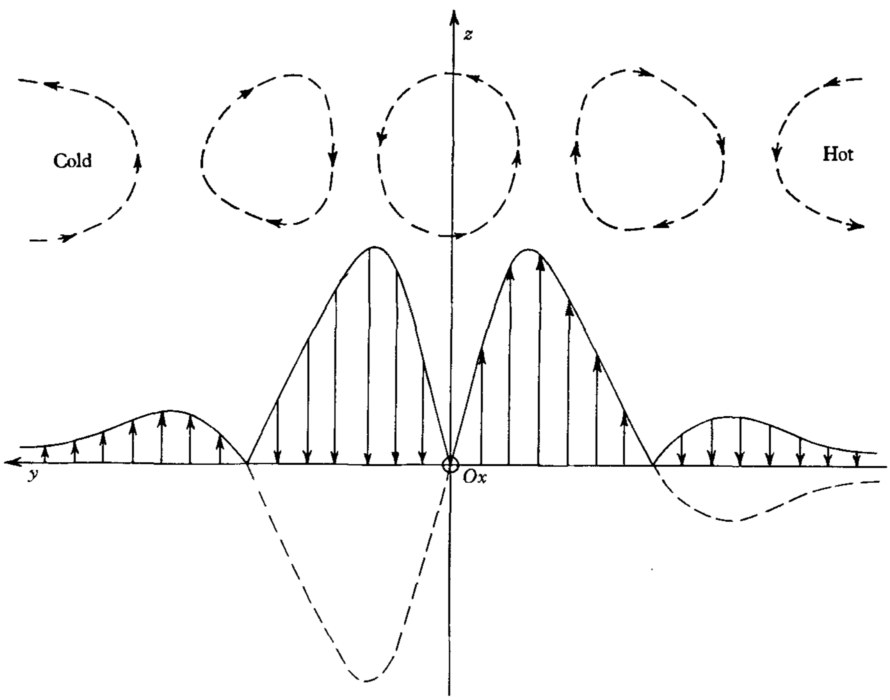

FIGURE 7. The cross-circulation in the second approximation, due to viscous forces proportional to $\partial^2 u/\partial y^2$.

If we can keep the viscous term in the form (54b), we have

$$\frac{\partial w_v}{\partial z} = \frac{8\nu\kappa}{\omega y_0^3}\frac{F_1(x)+G_1(z)}{F_1'(x)}\left(3\operatorname{sech}^4\frac{y-y_1(x)}{y_0}-\operatorname{sech}^2\frac{y-y_1(x)}{y_0}\right)\tanh\frac{y-y_1(x)}{y_0}$$
$$= \frac{4\nu}{\omega y_0}u\left(3\operatorname{sech}^2\frac{y-y_1(x)}{y_0}-1\right)\tanh\frac{y-y_1(x)}{y_0}.$$

This suggests a vertical velocity distribution and cross-circulation as shown in figure 7. This is not outstandingly different from that of figure 6, at least in the centre of the stream. In any case the two circulations must be superposed, and it seems likely that the small circulations away from the centre in figure 7 will be swamped by larger velocities of figure 6; if this is so, figure 6 again gives a qualitative picture of the circulation.

The validity of expanding the solution in powers of Θ is not discussed here; it is assumed that the expansion is possible, and the only justification given is that it gives a reasonable solution.

6. The corresponding problem in the atmosphere

Making assumptions similar to those of the first approximation in §2, and taking a similar choice of axes, the equations of motion and continuity in the atmosphere are

$$2\omega \sin\phi \,.\, u = -\partial h/\partial y,$$

$$2\omega \sin\phi \,.\, v = +\partial h/\partial x,$$

$$-\frac{1}{\rho} = \partial h/\partial p,$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y} = 0,$$

where $h = gz$ is the geopotential, and ϕ is the latitude of the origin of co-ordinates. In these equations, x, y and p are chosen as independent variables. The equation of state is now

$$p = R\rho T,$$

and the equation of heat transfer is still

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y} = \kappa\frac{\partial^2 T}{\partial y^2}.$$

The relation between T and p becomes

$$T = \frac{p}{R\rho} = -\frac{1}{R}p\frac{\partial h}{\partial p} = -\frac{1}{R}\frac{\partial h}{\partial q},$$

where $q = \log p$. The equations then reduce essentially to those of §2, and we have the equation for h:

$$-\frac{\partial h}{\partial y}\frac{\partial^2 h}{\partial x\,\partial q}+\frac{\partial h}{\partial x}\frac{\partial^2 h}{\partial y\,\partial q} = 2\omega\kappa\sin\phi\frac{\partial^3 h}{\partial y^2\,\partial q}.$$

A solution of exactly similar form for T, u, v is therefore obtained.

The solution can be applied immediately to a local jet. In this case, the variation of the function $y_0(x)$ of §4 should be retained; it will decrease as x increases at the entry to the jet, and increase with x at the exit. The extra term in v which occurs in equation (48) by taking $y_0'(x)$ non-zero is an odd function of $y-y_1(x)$; this corresponds to an inflow to $y = y_1(x)$ at the entrance, and an outflow at the exit, as would be expected. We note also that u is proportional to $1/[y_0(x)]^2$, and so increases as the stream narrows. The mechanism is similar to that of the confluence theory suggested by Clapp & Namias (1949).

It is more difficult to apply the same solution to a jet which circumvents the earth, as there are now no boundaries where it is reasonable to take a discontinuity. A solution of the equations which does not involve discontinuities would, however, be applicable to both cases.

7. Some difficulties of the present solution

Any complete solution of the problem of the jet-stream must give a satisfactory account of the transfer of angular momentum, and of heat, across the fluid. The present theory is at first sight unsatisfactory in both respects, but since the solution does not pretend to describe the flow completely, this is not a serious defect. It may be of interest to give a brief indication of the transport of these quantities across the stream.

The transfer of momentum behaves like the product uv, and is zero across the line $y = y_1(x)$, unless the function $F_2(x)$ is retained. Even in this case the transport is in opposite directions across the stream, according to whether the stream approaches the inner or the outer boundary, so that the mean with respect to x is zero. But the effect of the boundaries where the stream touches them is of considerable importance in practice, and the present theory takes no account of this.

No simple form for the transfer of heat across the stream exists, and it is impossible to interpret fully the appropriate expression to the solution of § 4. It is clear, however, that in the mean with respect to x, the transfer of heat across the line $y = y_1(x)$ is entirely by conduction. Again, the exchange of heat between the stream and the boundaries, which is omitted from the solution, is of major importance.

In order to describe the motion properly, then, we must find solutions for the parts of the jet-stream where it touches the boundaries, and also for the circulations in the remainder of the fluid. In principle this should be possible, but in practice it seems to be intractable.

The solution is of interest, however, because it does give a stream with the right kind of transverse variation in temperature and velocity. This suggests that the basic assumptions made in § 2 are valid, and a more general solution should be sought.

The author wishes to thank Prof. M. J. Lighthill for his continued interest in her work, and Prof. T. V. Davies for the benefit of several discussions during the preparation of this paper.

REFERENCES

Chandrasekhar, S. 1953 The instability of a layer of fluid which is heated from below and subject to Coriolis forces. *Proc. Roy. Soc.* A, **217**, 306–326.

Clapp, P. F. & Namias, J. 1949 Confluence theory of the high tropospheric jet-stream. *J. Met.* **6**, 330–336.

Davies, T. V. 1953 The forced flow of a rotating viscous liquid which is heated from below. *Phil. Trans.* A, **246**, 81–112.

Davies, T. V. 1956 The forced flow due to heating of a rotating liquid. *Phil. Trans.* A, **249**, 27–64.

Fultz, D. 1951 Experimental analogies to atmospheric motions. *Compendium of Meteorology*, pp. 1235–1248.

Hide, R. 1956 Fluid motion in the earth's core and some experiments on thermal convection in a rotating fluid. *Symposium on Geophysical Fluid Dynamics*. Baltimore, 1953.

HIDE, R. 1958 An experimental study of thermal convection in a rotating liquid. *Phil. Trans.* A, **250**, 441–478.

KUO, H.-L. 1954 Symmetrical disturbances in a thin layer of fluid subject to a horizontal temperature gradient and rotation. *J. Met.* **11**, 399–411.

LANCE, G. N. 1958 The effect of a vertical lapse rate of temperature on the spiral flow of a fluid in a heated rotating cylinder. *J. Fluid Mech.* **3**, 523.

LANCE, G. N. & DELAND, E. C. 1957 The steady, axially symmetric flow of a viscous fluid in a deep rotating cylinder heated from below. *Univ. Calif. Publ. Engng*, **5**, 159–176.

RIEHL, H. & FULTZ, D. 1957 Jet streams and long waves in a steady rotating-dishpan experiment: structure of the circulation. *Quart. J. Roy. Met. Soc.* **83**, 215–231.